

Particle Geodesics in the Kerr Spacetime

Jenna Van Dyke*

Valley Christian High School, San Jose, California

Mentor: Kyle Singh

Abstract

The Kerr spacetime is investigated. The singularity structure and horizon structure of the spacetime are explored. Two sets of equations of motion for a particle orbiting a Kerr black hole are derived: the first by the calculating nonzero Christoffel symbols of the metric and utilizing the geodesic equation; the second by dot products of Killing vectors and the four-velocity. The effective potential of the radial motion is found and analyzed in comparison with the Schwarzschild effective potential. The Kerr/CFT correspondence and its recent use in predicting polarimetric images of the supermassive black hole in the galaxy M87 is also discussed briefly. An overview of special relativity, general relativity, and orbits is provided in the appendix to serve as background information for the body of the paper. The Mathematica programs used for this paper can be found here: <https://bit.ly/3dlwYH7>.

*jenna.vandyke@warriorlife.net

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1 The Kerr Metric

The Kerr solution, derived by Roy Kerr in 1963, describes a rotating uncharged black hole. The line element is given by

$$ds^2 = - \left(1 - \frac{r_s r}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 - \frac{2r_s a r \sin^2 \theta}{\Sigma} d\phi dt + \left(r^2 + a^2 + \frac{r_s r a^2 \sin^2 \theta}{\Sigma}\right) \sin^2 \theta d\phi^2 \quad (1.1)$$

where $a = \frac{J}{M}$, $\Sigma = r^2 + a^2 \cos^2 \theta$, and $\Delta = r^2 - r_s r + a^2$. J is the spin angular momentum and $r_s = 2M$ is the Schwarzschild radius. The metric is given in Boyer-Lindquist coordinates (t, r, θ, ϕ) . It is t -independent (*stationary*) and ϕ -independent (*axially symmetric*).

When a , the spin angular momentum term, vanishes, the Kerr metric becomes the Schwarzschild metric¹. This makes sense—a Kerr black hole is the same as a rotating Schwarzschild black hole.

We know that the Kerr metric is a solution to Einstein's equation because the left (the Ricci tensor and Ricci scalar) and the right (the stress-energy tensor) sides of the equation both vanish. We will briefly show some of these calculations.

The complete list of nonzero components of the Riemann tensor can be found in [2]. To demonstrate the calculation we will only show $R_{r\theta\phi}^t$:

$$\begin{aligned} R_{r\theta\phi}^t &= \partial_\theta \Gamma_{\phi r}^t - \partial_\phi \Gamma_{\theta r}^t + \Gamma_{\theta\lambda}^t \Gamma_{\phi r}^\lambda - \Gamma_{\phi\lambda}^t \Gamma_{\theta r}^\lambda \\ &= 3\partial_\theta \Gamma_{\phi r}^t - \frac{r_s a^3 \sin^3 \theta \cos \theta}{\Sigma^3} \left[\frac{a^2 \cos^2 \theta (a^2 - r^2) - r^2 (a^2 + 3r^2)}{2\Delta} \left(\frac{r_s r}{\Sigma} - 1 \right) + r^2 \right] \end{aligned} \quad (1.2)$$

which can be shown to simplify to

$$R_{r\theta\phi}^t = \frac{r_s a \cos \theta (3r^2 - 3a^2 \cos^2 \theta)}{\Sigma^3} \quad (1.3)$$

¹It is important to note the difference between angular momentum and spin angular momentum. Both metrics contain factors of l , the angular momentum term. However, only the Kerr metric has spin angular momentum, given by J .

After finding the Riemann tensor, we can find the Ricci tensor from

$$R_{\mu\nu} = g^{\sigma\rho} R_{\sigma\mu\rho\nu} \quad (1.4)$$

Again we will show the calculation for only one component, $R_{\theta r}$:

$$R_{\theta r} = g^{\sigma\rho} R_{\sigma\theta\rho r} = g^{t\phi} R_{t\theta\phi r} + g^{\phi t} R_{\phi\theta t r} = \frac{r_s a \cos \theta (3r^2 - 3a^2 \cos^2 \theta)}{\Sigma^3} (g^{t\phi} - g^{\phi t}) = 0 \quad (1.5)$$

The Ricci tensor can be found to vanish completely. Since $R_{\mu\nu} = 0$,

$$R = g^{\mu\nu} R_{\mu\nu} = 0 \quad (1.6)$$

Since the Ricci tensor and Ricci scalar both vanish, so does the left side of Einstein's equation. Since the Kerr metric is a vacuum solution, the stress-energy tensor, and thus the right side of the equation, vanishes as well. Thus the Kerr metric is a solution to Einstein's equation.

2 Singularity Structure and Horizon Geometry

The metric has singularities when $\Sigma = 0$ and when $\Delta = 0$.

The $\Sigma = 0$ singularity is real, and it occurs when $r = 0$ and $\theta = \frac{\pi}{2}$. All the points at $r = 0$ form a disk, and the points at $r = 0$, $\theta = \frac{\pi}{2}$ make up the ring on the edge of that disk. Thus the Kerr singularity is fondly called the ring singularity. On another note, when $a = 0$, the singularity corresponds to the singularity of the Schwarzschild metric at $r = 0$.

The $\Delta = 0$ singularity occurs when $r_{\pm} = M \pm \sqrt{M^2 - a^2}$. However, it can be shown that this singularity is a coordinate singularity. Again, when $a = 0$, it becomes the Schwarzschild coordinate singularity at $r = 2M$.

If the points at $r_{\pm} = M \pm \sqrt{M^2 - a^2}$ are not real singularities, then what do they mean physically? Again going back to the Schwarzschild case, the point $r = 2M$ is the horizon radius that makes a Schwarzschild black hole; similarly, r_+ is the horizon radius that makes a Kerr black hole—this horizon is the *outer event horizon*—the

outermost ‘point of no return’. At radius r_- is the *inner event horizon*. Within it is again another region from which nothing can escape to outer regions.

There exists another region outside the outer event horizon that a particle can escape from if it has high enough velocity, but will be sucked into the black hole if it is stationary. This region is called the *ergosphere*. Inside the ergosphere the particle must move in the direction of the rotation. The ergosphere is defined by the outer ergosurface² at $r_E^+ = M + \sqrt{M^2 - a^2 \cos^2 \theta}$ and the outer event horizon.

Note that $r_E^+ \geq r_+ > r_- \geq r_E^-$ where $r_E^+ = r_+$ and $r_E^- = r_-$ at the poles ($\theta = 0$ and $\theta = \pi$).

Neither the horizons nor ergosurfaces are spherical. Because they are slices of Kerr spacetime, they have axial, but not spherical, symmetry, so they are squished vertically—a result of the rotation of the body.

3 Equations of Motion

We will first derive equations of motion for a particle’s orbit from Christoffel symbols and the geodesic equation.

The complete list of nonzero Christoffel symbols for Kerr can be found in [6]. To demonstrate the calculation, we show $\Gamma_{\theta\theta}^r$:

$$\Gamma_{\theta\theta}^r = \frac{1}{2}g^{r\alpha}(\partial_\theta g_{\theta\alpha} + \partial_\theta g_{\alpha\theta} - \partial_\alpha g_{\theta\theta}) \quad (3.1)$$

Summing over α ,

$$\frac{1}{2}g^{rr}(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}) = \frac{1}{2}g^{rr}(-\partial_r g_{\theta\theta}) = \frac{\Delta}{2\Sigma}(-2r) = -\frac{\Delta r}{\Sigma} \quad (3.2)$$

Plugging all nonzero Christoffel symbols into the geodesic equation, we get the following equations of motion:

²The inner ergosurface is at $r_E^- = M - \sqrt{M^2 - a^2 \cos^2 \theta}$. The ergosurfaces are also sometimes called the stationary limit surfaces.

$$\begin{aligned} & \frac{d^2 t}{d\tau^2} + \frac{2r_s(r^2 + a^2)(r^2 - a^2 \cos^2 \theta)}{2\Sigma^2 \Delta} \frac{dt}{d\tau} \frac{dr}{d\tau} + \frac{r_s r a^2 \sin \theta \cos \theta}{\Sigma^2} \left(-2 \frac{dt}{d\tau} \frac{d\theta}{d\tau} + 2a \sin^2 \theta \frac{d\theta}{d\tau} \frac{d\phi}{d\tau} \right) \\ & + \frac{2r_s a \sin^2 \theta [a^2 \cos^2 \theta (a^2 - r^2) - r^2 (a^2 - 3r^2)]}{2\Sigma^2 \Delta} \frac{dr}{d\tau} \frac{d\phi}{d\tau} = 0 \end{aligned} \quad (3.3)$$

The equations for r , θ , and ϕ can similarly be derived.

This is quite messy. Now let us utilize symmetry for a more efficient derivation of equations of motion.

In the Schwarzschild case, thanks to spherical symmetry, we are able to set $\theta = \frac{\pi}{2}$ and greatly simplify our equations. We would like to do the same here, but Kerr spacetime only has axial symmetry. The solution is to limit orbits to the equatorial plane. In reality Kerr orbits are more complex, but for our purposes we will only consider $\theta = \frac{\pi}{2}$. The metric in this case becomes

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \frac{r^2}{\Delta} dr^2 + \left(r^2 + a^2 + \frac{2Ma^2}{r} \right) d\phi^2 - \frac{4aM}{r} dt d\phi. \quad (3.4)$$

Because the metric is independent of t and ϕ , we have the Killing vectors

$$\xi_t = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.5a)$$

and

$$\xi_\phi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (3.5b)$$

Using these we can derive the following equations

$$- \left(1 - \frac{2M}{r} \right) \dot{t} - \frac{2aM}{r} \dot{\phi} = -e \quad (3.6a)$$

$$\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi} - \frac{2aM}{r}\dot{t} = \ell \quad (3.6b)$$

We can solve this system of equations for $\dot{t}$ and $\dot{\phi}$:

$$\dot{t} = \frac{1}{\Delta} \left[\left(r^2 + a^2 + \frac{2Ma^2}{r} \right) e - \frac{2Ma}{r} \ell \right] \quad (3.7a)$$

$$\dot{\phi} = \frac{1}{\Delta} \left[\left(1 - \frac{2M}{r} \right) \ell + \frac{2Ma}{r} e \right] \quad (3.7b)$$

We then plug these values into the equation $\mathbf{u} \cdot \mathbf{u} = -1$ to obtain

$$\begin{aligned} g_{\mu\nu} \mathbf{u}^\mu \mathbf{u}^\nu &= g_{tt} \mathbf{u}^t \mathbf{u}^t + g_{rr} \mathbf{u}^r \mathbf{u}^r + g_{\theta\theta} \mathbf{u}^\theta \mathbf{u}^\theta + g_{\phi\phi} \mathbf{u}^\phi \mathbf{u}^\phi + g_{\phi t} \mathbf{u}^\phi \mathbf{u}^t + g_{t\phi} \mathbf{u}^t \mathbf{u}^\phi \\ &= - \left(1 - \frac{r_s r}{\Sigma} \right) \dot{t}^2 + \frac{\Sigma}{\Delta} \dot{r}^2 + \left(r^2 + a^2 + \frac{r_s a^2 \sin^2 \theta}{\Sigma} \right) \dot{\phi}^2 - \frac{2r_s r a \sin^2 \theta}{\Sigma} \dot{t} \dot{\phi} = -1 \end{aligned} \quad (3.8)$$

It can be shown that (3.8) simplifies to

$$\frac{1}{2} \dot{r}^2 + V_K = \frac{e^2 - 1}{2} \quad (3.9)$$

where

$$V_K = -\frac{M}{r} + \frac{\ell^2 - a^2(e^2 - 1)}{2r^2} - \frac{M(\ell - ae)^2}{r^3} \quad (3.10)$$

is the effective potential.

4 Kerr Orbits

Let us now analyze the effective potential for Kerr and compare it with the Schwarzschild effective potential (Figure 1).

- Consistent with the pattern we have seen thus far, when $a = 0$, the Kerr potential becomes the Schwarzschild potential.
- Due to the presence of spin angular momentum, the Kerr potential always peaks higher than the Schwarzschild potential. More angular momentum corresponds to a greater difference between the two. The difference is further accentuated by the factors of e in the Kerr potential, which are absent from the Schwarzschild potential.
- For the current values of fixed parameters³ (M , ℓ , a , and e), the Kerr potential has a deeper well further to the left, indicating that E_0 and r_0 , the energy and radius of a circular orbit respectively, are less for the rotating black hole than the static black hole. However, this is not true in every case. If there is enough spin angular momentum, the Schwarzschild well will eventually move below and to the left of the Kerr well. Increasing the eccentricity causes Kerr to move even farther below Schwarzschild.
- By setting $V_S > V_K$ and $M = 1$, we can obtain the inequality

$$r > -\frac{e}{a}(ae - 2\ell) \quad (4.1)$$

which tells us the value of r for which the Schwarzschild potential is greater than the Kerr potential. The physical reason for this particular rule is unclear, but it can be used for numerics.

Figure 2 shows the accumulated angles from the innermost turning point as a function of position along the orbit.

5 Black Hole Imaging Using the Kerr/CFT Correspondence

The Event Horizon Telescope (EHT) will soon release images of the supermassive black hole in the center of the galaxy M87. One of the results will be a polarimetric image showing the polarization of the incident light, which will indicate the dynamics of the black hole's surrounding region.

In general, such imaging is extremely complicated due to the black hole's complex surrounding environment. Various astrophysical effects and unknown properties of

³A note on parameters: We set $\varepsilon = -0.01$ so that we have elliptical orbits.

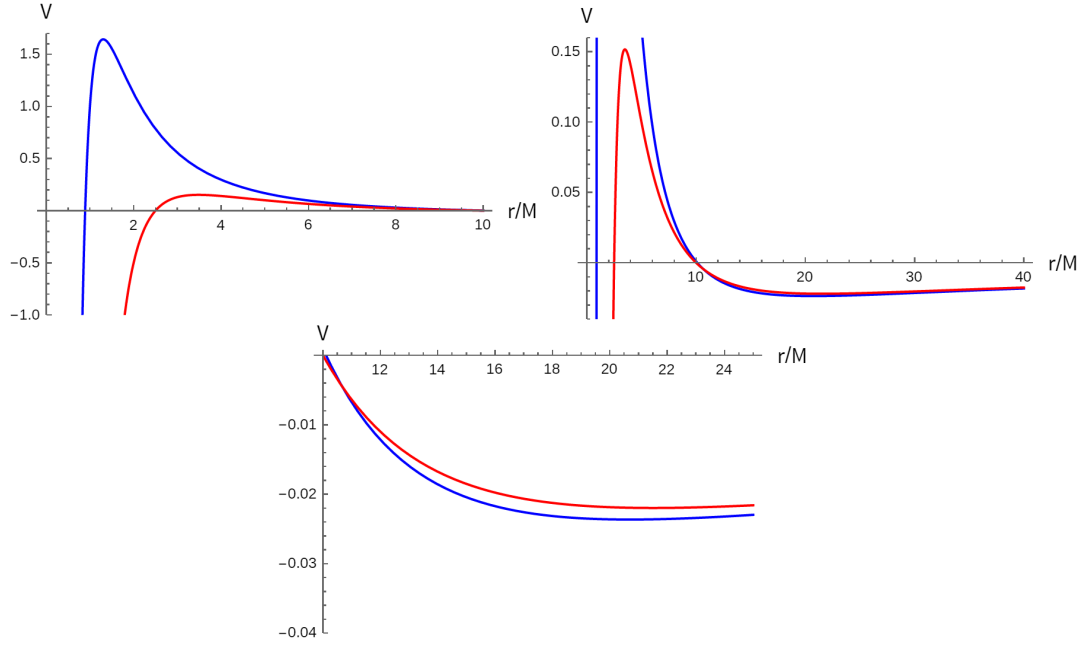


Figure 1: The Kerr effective potential (blue) versus the Schwarzschild effective potential (red). On the top left and right the Kerr and Schwarzschild peaks, respectively, are shown; on the bottom both wells are shown.

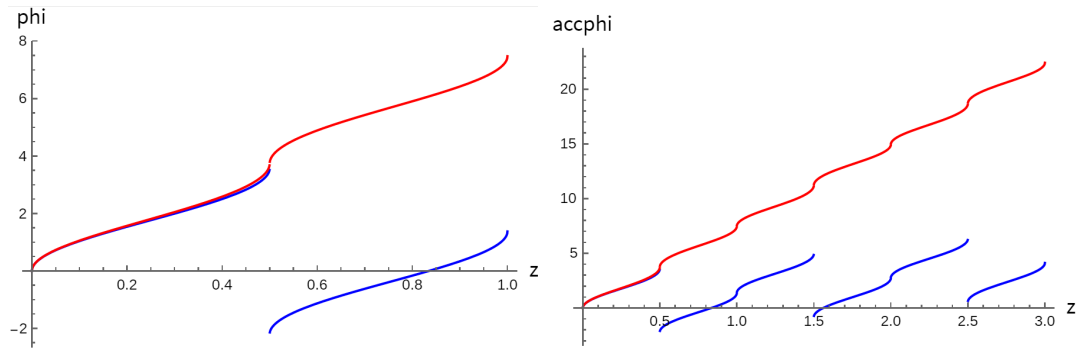


Figure 2: The accumulated angles as a function of position along the orbit, where $z = 1$ is the end of the first orbit, $z = 2$ is the end of the second orbit, etc.

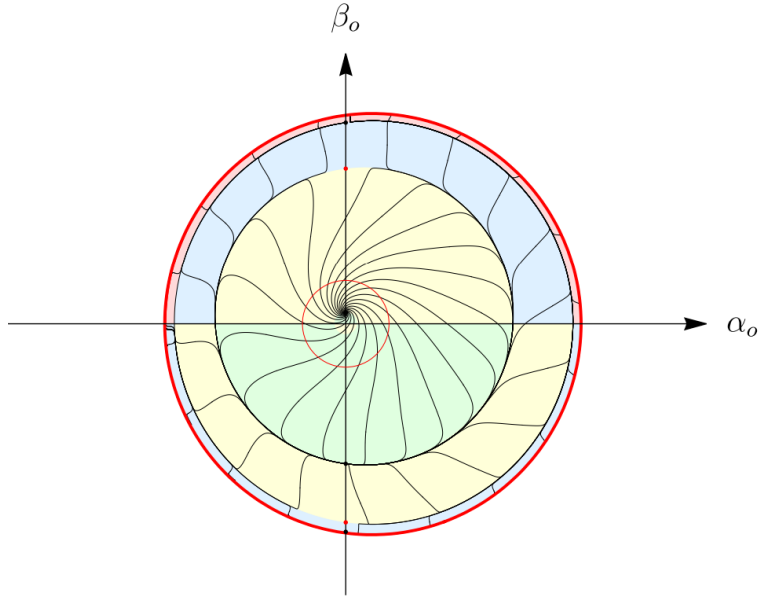


Figure 3: The predicted polarimetric image for the black hole in the galaxy M87 at an angle of 15° from the pole, as observed by the EHT. The black lines represent the integral curves of the electromagnetic field. The inner and outer red circles correspond to the event horizon and the edge of the black hole's shadow, respectively. The red dots along the vertical axis indicate the poles. The different colored regions correspond to the number m of angular turning points of the geodesic connecting the source of the emissions to the EHT. The colors for $m = 0, 1, 2, 3$ light are green, yellow, blue, and red, respectively.

the surrounding matter require numerical simulations. However, M87 is a rapidly spinning (Kerr) black hole. According to general relativity, such spinning black holes have conformal symmetry⁴ (scale invariance) in their near-horizon regions, greatly simplifying the calculations required for the imaging and allowing for analytic simulation.

In [9] the conformal correspondence is used to predict the polarimetric images of M87’s black hole. The results are shown in Figure 3.

6 Acknowledgements

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A Fundamentals of Relativity

Special and general relativity, both published by Einstein in 1905 and 1916 respectively, work together to produce our current understanding of the structure of space-time and gravity.

A.1 Special Relativity

Before we discuss special relativistic effects, let us think about what physics is like from a Newtonian perspective. If two people observe any physical event, such as a ball rolling down a hill, they will measure the same time interval from when it started at the top to when it stopped at the bottom. This concept that time is not relative is a given in our everyday life—something that we do not even think about. But it is not a guarantee in special relativity.

At its core, special relativity is a theory of the nature of space and time. It is based on two postulates:

1. The laws of physics are the same in all inertial frames of reference.⁵
2. The speed of light in a vacuum is always constant at $c = 299,792,458$ m/s.

⁴This relationship is known as the Kerr/CFT correspondence.

⁵An inertial frame of reference is a frame of reference that is not accelerating.

A.1.1 Time Dilation

We can use these ideas in a thought experiment to better understand the strange implications of special relativity. Consider two observers standing face-to-face, each holding a mirror. A ball of light is fired between the two mirrors, and it bounces back and forth. The mirrors are like clocks, ticking each time they receive the ball.

While both observers are stationary, they measure equal time intervals. However, if observer A was moving with constant velocity to the side then observer B would see the ball bouncing back and forth on a diagonal trajectory, a longer trajectory than before. Thus, B would see A's light clock ticking slower; they would observe longer time intervals. This phenomenon is called *time dilation*. A would not see any difference at all; from their perspective, their light clock would tick the exact same as before. A's measurement is called the *proper time*, τ . The difference between A and B's measurements is given by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (\text{A.1})$$

where v is A's velocity and γ is known as the Lorentz factor. We also have

$$t = \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (\text{A.2})$$

where t is the time measured by B.

Time dilation results in length contraction (and length contraction results in time dilation) by the formula $d = rt$. If there is time dilation, then there must also be length contraction to keep the rate c constant (as required by Einstein's second postulate).

A.1.2 Lorentz Transformations

Lorentz transformations are another crucial concept in special relativity. We said that the Lorentz factor γ accounts for the differences due to time dilation and length contraction, so we can use (A.1) to write Lorentz transformations:

$$x' = \gamma(x - vt) \quad (\text{A.3a})$$

$$y' = y \quad (\text{A.3b})$$

$$z' = z \quad (\text{A.3c})$$

$$t' = \gamma \left(t - \frac{vx}{c^2} \right) \quad (\text{A.3d})$$

A.1.3 The Metric and Summation Convention

In special relativity, spacetime is composed of three spatial dimensions and one temporal dimension. An individual point in spacetime is called an *event*, and the path a particle takes across a set of events—its trajectory—is called the *worldline*. With the coordinates (t, x, y, z) , the equation for the distance between two events, called the *spacetime interval*, is given by

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \quad (\text{A.4})$$

Notice how this equation resembles the distance formula in three dimensions, the Pythagorean theorem. The spacetime interval simply adds a temporal dimension with a factor of c^6 . From this point forward we will set $c = 1$ for simplicity.

The infinitesimal spacetime interval, called the *line element*, is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad (\text{A.5})$$

We can write the spacetime interval more compactly as a 4×4 matrix called the *metric*:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A.6})$$

where $\mu, \nu = 0, 1, 2, 3 = t, x, y, z$.⁷

To write the metric from a line element, we place the coefficients of each term in the line element into the corresponding entry of the matrix. In (A.5), the coefficient of

⁶The constant c acts as a conversion factor between space and time.

⁷Typically Latin indices run from one to three and Greek indices run from zero to three.

the term $-dt^2$ is -1 , so we enter -1 into g_{tt} . The coefficients of the rest of the terms are all 1, so we enter 1 into g_{xx} , g_{yy} , and g_{zz} . The rest of the entries are 0.

We can think of the metric as the unit vectors for our spacetime. (A.6), the *Minkowski metric*, is simply the three Euclidean spatial coordinates plus time, hence the spatial components of the metric (the Euclidean unit vectors) are all equal to 1.

The metric is a *tensor*. Tensors are multidimensional arrays of numbers that obey certain transformation laws. Tensors of different ranks are represented by different objects. A rank-0 tensor is simply a scalar, a rank-1 tensor is a vector, a rank-2 tensor is a two-dimensional array, and so on.

Tensors have *indices* as superscripts and subscripts; the number of indices indicates the rank of the tensor. For example, a (2,2) tensor $A_{\sigma\rho}^{\mu\nu}$ is a rank-4 tensor.

There are a few basic tensor operations we will use throughout the paper:

- Summation convention is the notation used for tensors. The sum $\sum_{i=1}^3 a_i b^i$ can be evaluated in summation convention as

$$a_i b^i = a_1 b^1 + a_2 b^2 + a_3 b^3 \quad (\text{A.7})$$

We simply remove the summation symbol.

Here $i = 1, 2, 3$.

In summation convention, indices repeated in a single term, called *dummy indices*, are summed over.

- Contraction is a unitary operation in which we set an upper index equal to a lower index, reducing the tensor's rank from (a, b) to $(a - 1, b - 1)$. Suppose we have our tensor from earlier $A_{\sigma\rho}^{\mu\nu}$. If we replace both ν and σ with α , we get the contraction

$$A_{\alpha\rho}^{\mu\alpha} = A_{\rho}^{\mu} \quad (\text{A.8})$$

- Another operation is the raising and lowering of indices, changing a tensor's rank from (a, b) to $(a + 1, b - 1)$ (raising) or $(a - 1, b + 1)$ (lowering). This is done by multiplying the tensor by the metric and then contracting:

$$A_{\mu} = g_{\mu\nu} A^{\nu} \quad (\text{A.9})$$

A.1.4 Four-Vectors

Now that we are working in four-dimensional spacetime, vectors have four components instead of three and are known as *four-vectors*. Some examples of four-vectors are:

- Four-position $\mathbf{x}^\mu = (t, x, y, z)$
- Four-velocity $\mathbf{u}^\mu = (\dot{t}, \dot{x}, \dot{y}, \dot{z}) = (\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau})$
- Four-momentum $\mathbf{p}^\mu = m\mathbf{u}^\mu$

Four-vectors transform under Lorentz transformations:

$$\mathbf{x}^\mu = \Lambda_{\nu}^{\mu'} \mathbf{x}^\nu \quad (\text{A.10})$$

where Λ is the Lorentz transformation matrix. Λ applies the same transformations as described in (A.3); the transformations are simply encoded in a matrix.

A.2 General Relativity

In the previous section we discussed special relativity; we described physics, particularly the structure of spacetime, in a relativistic framework and learned some basic mathematical tools. We now expand on those ideas by giving a brief introduction to general relativity, the theory which explains what happens when we add matter and energy to the mix: gravitation!

Imagine a physicist inside a box conducting an experiment to measure acceleration due to gravity. In their experiment, they might measure the effects of an external gravitational field, but it is also possible for the acceleration to be due to the box's acceleration, not the acceleration due to gravity. It is impossible to measure the acceleration due to gravity using local experiments. This is called the *Einstein Equivalence Principle*, and it motivates the fundamental idea of general relativity.

If acceleration due to gravity cannot be reliably measured, then we cannot define gravity as a force either. Rather, we consider gravitation an effect of the curvature of spacetime. The presence of matter and energy curves spacetime, creating gravity.

General relativity describes the motion of particles in curved spacetime. Free particles naturally take the path of shortest distance, a straight line, called a *geodesic*. You can think of it as the path taken if you were to continuously walk forward on the

surface. Even on a curved surface, such a path is a straight line, although it might not look like what we would consider to be a straight line.

When calculating geodesics in curved space, we cannot use a simple distance formula like we would in flat space; instead we must use the *geodesic equation*:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0 \quad (\text{A.11})$$

where $\Gamma_{\rho\sigma}^\mu$ is an object known as the *Christoffel symbol*⁸, given by

$$\Gamma_{\rho\sigma}^\mu = \frac{1}{2} g^{\mu\nu} (\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\rho\sigma}) \quad (\text{A.12})$$

Note that the lower indices are symmetric, meaning $\Gamma_{\alpha\beta}^\lambda = \Gamma_{\beta\alpha}^\lambda$. This will be important later on.

Also notice that the Christoffel symbol is just fluctuations (derivatives) of the metric. Since the motion of objects in a particular spacetime is influenced by the structure of that spacetime, we can imagine that Christoffel symbols might have something to do with curvature as well—that fluctuations of the fundamental object of the structure of spacetime describe curvature—this intuition is correct. The mathematical description of curvature is encoded in the *Riemann tensor*:

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda \quad (\text{A.13})$$

The Riemann tensor has several useful properties: It is antisymmetric on the pair of the first two indices and the pair of the last two indices, and it is symmetric on those pairs:

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\nu\mu} = R_{\mu\nu\rho\sigma} \quad (\text{A.14})$$

If we take a contraction of the Riemann tensor, we get the *Ricci tensor*:

$$R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda = g^{\rho\sigma} R_{\rho\mu\sigma\nu} \quad (\text{A.15})$$

The Ricci tensor is also symmetric ($R_{\mu\nu} = R_{\nu\mu}$).

⁸Despite its notation, the Christoffel symbol is not a tensor, rather an “object” or “symbol.”

The trace, or the sum of the diagonal components, of the Ricci tensor is the *Ricci scalar*, also called the *curvature scalar*:

$$R = R^\mu_\mu = g^{\mu\nu} R_{\mu\nu} \quad (\text{A.16})$$

With all these building blocks we can write Einstein's equation for general relativity, which describes the relationship between matter and the curvature of spacetime. It is given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (\text{A.17})$$

where G is Newton's gravitational constant and $T_{\mu\nu}$ is the *stress-energy tensor*. The stress-energy tensor holds information about the energy, pressure, and stress; it describes all of the matter in the spacetime. The four-momentum $\mathbf{p}^\mu$ is for an individual particle; it is an insufficient description of the energy of an extended system, which is where the stress-energy tensor comes in. The simplest stress-energy tensor is for a perfect fluid, in which case $T_{\mu\nu}$ takes the form

$$\begin{pmatrix} \rho_E & 0 & 0 & 0 \\ 0 & P_x & 0 & 0 \\ 0 & 0 & P_y & 0 \\ 0 & 0 & 0 & P_z \end{pmatrix} \quad (\text{A.18})$$

where ρ_E is the energy density and P_i contains the various components of pressure.

A.2.1 Black Holes

Many metrics are solutions to Einstein's equation. The first non-trivial exact solution was derived in 1916 by Karl Schwarzschild. The line element for his solution is given by

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad (\text{A.19})$$

so the metric is

$$g_{\mu\nu} = \begin{pmatrix} -(1 - \frac{2M}{r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (\text{A.20})$$

where M is the object's mass. Note that we are now using spherical coordinates (t, r, θ, ϕ) .

The Schwarzschild solution describes the spacetime around a non-rotating, uncharged black hole. At radius $r = 2M$ is the “point of no return”, or *event horizon*; the radius of the event horizon is called the *Schwarzschild radius*⁹. An observer outside the black hole cannot see past the event horizon, since the light inside cannot escape to deliver that information to them. Although $r = 2M$ makes g_{rr} undefined, there is not a singularity at these points in spacetime; it is only undefined as a result of the coordinate system we have chosen.

There is, however, a true singularity, regardless of our chosen coordinate system, at $r = 0$, causing g_{tt} and g_{rr} to be undefined. Physically, a singularity means that there is infinite curvature at that point inside the black hole.

The Schwarzschild solution is a vacuum solution, meaning its Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ (the left side of Einstein's equation) vanishes. Since there no curvature, there is no matter to curve the spacetime. Thus, the stress-energy tensor $T_{\mu\nu}$ vanishes as well.

A.2.2 Equations of Motion for the Schwarzschild Metric

There are two ways we can obtain equations of motion, which describe the motion of a particle around a Schwarzschild black hole:

1. Find all nonzero Christoffel symbols of the metric and plug them into the geodesic equation.
2. Use symmetries of the metric and the equation $\mathbf{u} \cdot \mathbf{u} = g_{\mu\nu} \mathbf{u}^\mu \mathbf{u}^\nu = -1$. to derive equations of motion.

Let us try the first method. The complete list of nonzero Christoffel symbols for Schwarzschild can be found in [6] but to demonstrate the calculation we show $\Gamma_{r\phi}^\phi$:

⁹The Schwarzschild radius is actually at $r = 2GM$, but since we are using “natural units”, $G = c = 1$. Similarly, every $2M$ term in the Schwarzschild metric is actually $2GM$.

$$\Gamma_{r\phi}^\phi = \Gamma_{\phi r}^\phi = \frac{1}{2}g^{\phi\alpha}(\partial_r g_{\phi\alpha} + \partial_\phi g_{\alpha r} - \partial_\alpha g_{r\phi}) \quad (\text{A.21})$$

Summing over α ,

$$\frac{1}{2}g^{\phi\phi}(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}) = \frac{1}{2}g^{\phi\phi}(\partial_r g_{\phi\phi}) = \frac{1}{2r^2 \sin^2 \theta}(2r \sin^2 \theta) = \frac{1}{r} \quad (\text{A.22})$$

After finding all nonzero Christoffel symbols, we plug them into the geodesic equation, which again is

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0 \quad (\text{A.23})$$

We will have four equations of motion, one for each dimension. Some equations have several Christoffel symbols; for example, the ϕ equation has $\Gamma_{\phi\theta}^\phi$ and $\Gamma_{\phi r}^\phi$. In that case we simply add another term for each Christoffel symbol. We get

$$\frac{d^2 t}{d\tau^2} + \frac{M}{r(r-2M)} \frac{dt}{d\tau} \frac{dr}{d\tau} = 0 \quad (\text{A.24})$$

The equations for r , θ , and ϕ can be similarly derived.

Now let us try the second method, which we will see is much simpler and less tedious. Since we have four dynamic variables, we (presumably) want four equations of motion, each requiring a symmetry to derive.

A physical system is symmetrical when a transformation is applied, and nothing changes about the physics. The Schwarzschild metric is invariant under transformations of t and ϕ ; we can immediately tell that this is true by the fact that the components of the metric are independent of the variables t and ϕ —if t or ϕ change, the metric is unaffected.

Killing vectors express symmetries of the metric. For the Schwarzschild metric the Killing vectors are

$$\xi_t = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (\text{A.25a})$$

and

$$\xi_\phi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (\text{A.25b})$$

We will use these Killing vectors for the cases $\mu = 0$ and $\mu = 3$ (t and ϕ respectively). For $\mu = 2$ (θ), thanks to spherical symmetry, we can set θ to any constant we would like. The most convenient choice is $\theta = \frac{\pi}{2}$, so that $\dot{\theta} = \ddot{\theta} = 0$ and $\sin^2 \theta = 1$. Because the derivatives vanish we will not have an equation of motion for θ .

We cannot use symmetry for $\mu = 1$ (r), so we will use another trick involving the four-velocity $\mathbf{u}^\mu$.

Normalization of the four-velocity (multiplying it by itself) produces

$$\mathbf{u} \cdot \mathbf{u} = g_{\mu\nu} \mathbf{u}^\mu \mathbf{u}^\nu = -1 \quad (\text{A.26})$$

We will use this result for the radial equation of motion.

Now we have the tools to derive the equations. Beginning with $\mu = 0$ and $\mu = 3$, we dot $\mathbf{u}^\mu$ with the Killing vectors:

$$\begin{aligned} \mathbf{u} \cdot \xi_t &= g_{\mu\nu} \mathbf{u}^\mu \xi_t^\nu = g_{00} \mathbf{u}^0 \xi_t^0 + g_{11} \mathbf{u}^1 \xi_t^1 + g_{22} \mathbf{u}^2 \xi_t^2 + g_{33} \mathbf{u}^3 \xi_t^3 \\ &= - \left(1 - \frac{2M}{r} \right) \dot{t} = -e \end{aligned} \quad (\text{A.27})$$

and

$$\begin{aligned} \mathbf{u} \cdot \xi_\phi &= g_{\mu\nu} \mathbf{u}^\mu \xi_\phi^\nu = g_{00} \mathbf{u}^0 \xi_\phi^0 + g_{11} \mathbf{u}^1 \xi_\phi^1 + g_{22} \mathbf{u}^2 \xi_\phi^2 + g_{33} \mathbf{u}^3 \xi_\phi^3 \\ &= r^2 \dot{\phi} = \ell \end{aligned} \quad (\text{A.28})$$

where ℓ is the angular momentum. For the radial equation,

$$\begin{aligned} \mathbf{u} \cdot \mathbf{u} &= g_{\mu\nu} \mathbf{u}^\mu \mathbf{u}^\nu = g_{00} \mathbf{u}^0 \mathbf{u}^0 + g_{11} \mathbf{u}^1 \mathbf{u}^1 + g_{22} \mathbf{u}^2 \mathbf{u}^2 + g_{33} \mathbf{u}^3 \mathbf{u}^3 \\ &= - \left(1 - \frac{2M}{r} \right) \dot{t}^2 + \left(1 - \frac{2M}{r} \right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 = -1 \end{aligned} \quad (\text{A.29})$$

It can be shown that (A.30) can be rewritten as

$$\frac{1}{2}\dot{r}^2 + V_S = \varepsilon \quad (\text{A.30})$$

where

$$V_S = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} \quad (\text{A.31})$$

is the effective potential,

$$\varepsilon = \frac{e^2 - 1}{2} \quad (\text{A.32})$$

is the orbital energy, and $e = (1 - \frac{2M}{r})\dot{t}$ is the eccentricity of the orbit.

Although the equations obtained from geodesics look quite different to those obtained from symmetry, they are the same. The equations from geodesics are of second order, so we can integrate them to obtain the equations from symmetry. Similarly, we can differentiate the equations from symmetry to obtain the equations from geodesics.

B Introducing Orbits

We can describe orbits in terms of their kinetic and potential energy,

$$\frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) = \varepsilon \quad (\text{B.1})$$

where $V_{\text{eff}}(r)$ is an effective potential such as (A.31). Classically a factor of the orbiting object's mass m is included in the kinetic term. However, in this paper we are describing particles' orbits relativistically, in which case we do not consider factors of m .

Let us consider the Schwarzschild effective potential (A.31) and see what it tells us about a particle's orbit around a Schwarzschild black hole. On the effective potential graph (Figure 4), different energy values correspond to different types of orbits: the energy at the local minimum corresponds to the energy of a circular orbit; the energies

between the minimum and the x -axis are for elliptical orbits; $\varepsilon = 0$ is for parabolic orbits; and $\varepsilon > 0$ is for hyperbolic orbits.

The relationship between energy and eccentricity is described by (A.32), which we can use to prove which energy values correspond to each orbital shape, since eccentricities of $e = 0$, $0 < e < 1$, $e = 1$, and $e > 1$ correspond to circles, ellipses, parabolas, and hyperbolas, respectively. Circular and elliptical orbits are bound, and parabolic and hyperbolic orbits are unbound, meaning the particle will not remain in orbit.

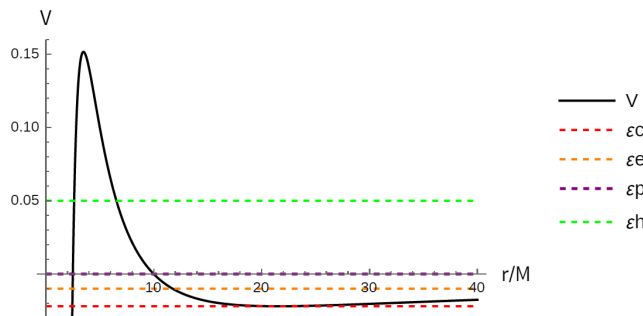


Figure 4: The Schwarzschild effective potential showing possible energies for circular, elliptical, parabolic, and hyperbolic orbits.

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